Rigidity of the unstable foliation

Sergi Burniol

IMERL, CMAT, Universidad de la República

sergi.burniol@gmail.com

UNIVERSIDAD DE LA REPÚBLICA URUGUAY

Introduction

Let f_t be an Anosov flow on a closed manifold M and let $\mathcal{W}^s_f$ and $\mathcal{W}^u_f$ denote the stable and the unstable foliations of f_t.

Definition. Given two Anosov flows f_t and g_t, the foliations $\mathcal{W}^u_f$ and $\mathcal{W}^u_g$ are *equivalent* if there exists a homeomorphism ϕ that maps one to the other.

Question. (Rigidity) If $\mathcal{W}^u_f$ and $\mathcal{W}^u_g$ are equivalent, what can be said about f_t and g_t?

Ratner established the first rigidity result for geodesic flows of hyperbolic surfaces in the measurable setting [6].

Theorem. **[3, 1]** *Let $(S, g), (S', g')$ be two negatively curved surfaces. If the unstable foliations on the unit tangent bundles T^1S and T^1S' are equivalent, then the two surfaces (S, g) and (S', g') are homothetic.*

Main Theorem. **[2]** *Let f_t, g_t be two C^1 transitive Anosov flows on a closed 3-manifold M. If the unstable foliations $\mathcal{W}^u_f$ and $\mathcal{W}^u_g$ are equivalent, then the flows f_t and $g_{\lambda t}$ are topologically conjugate for some $\lambda > 0$.*

Conjugacy is much stronger than orbit equivalence: there's an infinite dimensional moduli of nonequivalent unstable foliations!

Question. Can topological conjugacy be improved? (work in progress of Gogolev – Leguil – Rodriguez-Hertz)

Plante's alternative

Theorem. **[5]** *Let $f_t : M \to M$ be a transitive Anosov flow. Then one and only one of the following is true:*

1. *f_t is topologically mixing and the foliations $\mathcal{W}^s$ and $\mathcal{W}^u$ are minimal.*
2. *f_t is the constant time suspension of an Anosov diffeomorphism on a compact C^1 submanifold of codimension one in M.*

For a codimension 1 Anosov flow, if $\mathcal{W}^s$ and $\mathcal{W}^u$ are *jointly integrable*, then f_t is a constant time suspension of an Anosov diffeomorphism.

The two cases are treated separately in the proof of the Main Theorem.

Topologically mixing Anosov flows

Assume that f_t and g_t are topologically mixing. Let $\phi : M \to M$ be a homeomorphism which maps $\mathcal{W}^u_f$ to $\mathcal{W}^u_g$ i.e. for every $x \in M$,

$$\phi(W^u_f(x)) = W^u_g(\phi(x)).$$

Step 1. *The center unstable foliation $\mathcal{W}^{cu}_f$ is mapped by ϕ to the center unstable foliation $\mathcal{W}^{cu}_g$.*

Remark. Previous works use properties specific to geodesic flows.

Proof. We show that if ϕ does not map $\mathcal{W}^{cu}_f$ to $\mathcal{W}^{cu}_g$ then $\mathcal{W}^s_g$ and $\mathcal{W}^u_g$ are jointly integrable.

Fact: Topological transversality between $W^{cu}_g(\phi(x))$ and $\phi(W^{cu}_f(x))$.

The green segment is formed by the projections of $\phi(f_t(x))$, $t \in [0, \varepsilon]$ to $\mathcal{W}^{cs}_g(\phi(x))$. The blue surface formed by long unstable segments of points in the green segment. The red surface is formed by short stable segments from points on $W^u_g(\phi(x))$.

Red and blue surfaces are separated along g_t direction.

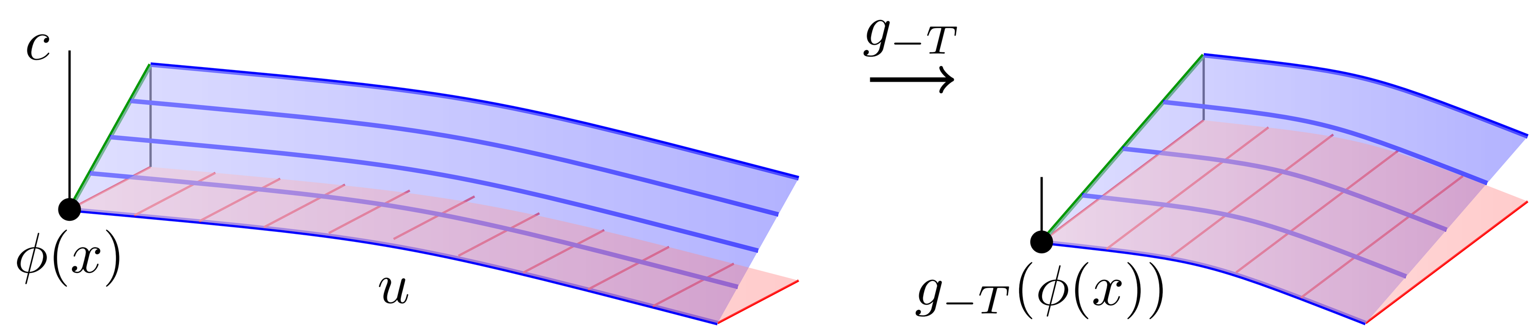

Figure 1: Red and blue surfaces and their image by g_{-T}

We apply g_{-T} with T big until the stable segments have size 1. Obtain two very close surfaces red and blue of approximate size 1. This couple of surfaces accumulate to surface foliated by both stable and unstable segments.

Step 2. *There exists a constant $\lambda > 0$ such that for all $x \in M$ and all $t \in \mathbb{R}$,*

$$\phi(W^u_f(f_t(x))) = g_{\lambda t}(W^u_g(\phi(x))).$$

Proof. By the previous step $\phi(W^u_f(f_t(x))) = g_{T(x,t)}(W^u_g(\phi(x)))$. Minimality of $\mathcal{W}^u_f$ and additivity of T in t imply $T(x, t) = \lambda t$.

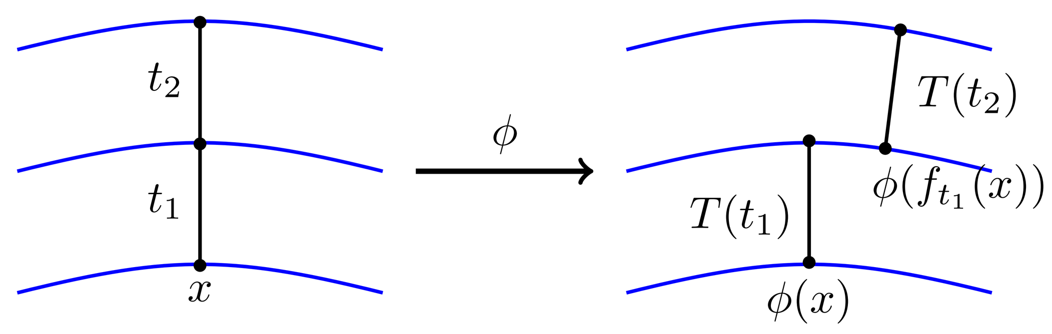

Figure 2: Image of the center unstable

Step 3. *Construct a homeomorphism $\psi : M \to M$ that conjugates f_t to $g_{\lambda t}$.*

Proof. Show that the maps

$$\phi_T = g_{-\lambda T} \circ \phi \circ f_T$$

converge to a map ψ when $T \to +\infty$. It automatically satisfies $\psi \circ f_t = g_{\lambda t} \circ \psi$.

Remark. In the case of geodesic flows, flow conjugacy plus Otal's result [4] shows that the metrics are homothetic.

The case of constant time suspensions

Assume that f_t and g_t are constant time suspensions of Anosov diffeomorphisms F and G.

By a topological argument, M being the suspension of both F and G implies that they are conjugate. Hence f_t and g_t are conjugate after constant time change.

References

[1] Ryuji Abe. Geometric approach to rigidity of horocycles. *Tokyo J. Math.*, 18(2):271–283, 1995.

[2] Sergi Burniol. Rigidity of the unstable foliation (preprint). 2025.

[3] Brian Marcus. Topological conjugacy of horocycle flows. *Amer. J. Math.*, 105(3):623–632, 1983.

[4] Jean-Pierre Otal. Le spectre marqué des longueurs des surfaces à courbure négative. *Ann. of Math. (2)*, 131(1):151–162, 1990.

[5] Joseph F. Plante. Anosov flows. *Amer. J. Math.*, 94:729–754, 1972.

[6] Marina Ratner. Rigidity of horocycle flows. *Ann. of Math. (2)*, 115(3):597–614, 1982.